Introduction to modelling and simulation of the Dice Game

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The Dice Game is a turn-based simulation of a system of dependent steps that are subject to statistical fluctuation. Processing at each turn is batched so that any early-release input from a previous player is not available until the next turn. In the base game, random variation is introduced using a standard six-sided die. The following introduces some modelling and simulation techniques that can be applied to the game.

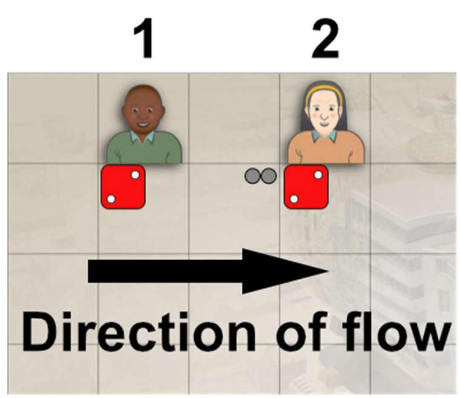
Some definitions:

Players are numbered 1, 2, …, N, with player 1 introducing counters into the system.
Let $Q_{i,i+1}$ be the number of counters between player i and player $i+1$ ($i < N$)
When the game is first set up $Q_{i,i+1} = Q_0 > 0 \quad \forall\, Q_{i,i+1}$.
Let $P_{i,i+1}(N)$ be the probability that $Q_{i,i+1}$ has exactly N counters.
Let R_i be the die roll value of player i on a particular turn.

Exercise 1: *Show that the minimum value of $Q_{i,i+1}$ is 1 for all rounds played.*

Simplifying the problem.

To begin the analysis, it is useful to simplify the base game, which has 10 players and dice that range from 1 to 6, to something more manageable. In the first case, let's consider dice that can roll a 1 or a 2, so we only have two outcomes to consider for each die, and also consider only the first two players and the first counter pile $Q_{1,2}$, which will be initialised to two counters.



There are four possible roll outcomes

$Q_{1,2}$ at the beginning of the turn	R_1	R_2	$Q_{1,2}$ at the end of the turn
2	1	1	2
2	2	1	3
2	1	2	1
2	2	2	2

From the above it is easy to construct the probability of outcomes after the first round as

$$P_{1,2}(1) = \frac{1}{4}$$

$$P_{1,2}(2) = \frac{1}{2}$$

$$P_{1,2}(3) = \frac{1}{4}$$

The above states that 50% of the time $Q_{1,2}$ won't change, 25% of the time it will reduce by 1 and 25% of the time it will increase by 1. For the case where it increases by 1 to 3, the following outcomes are available in the next round

$Q_{1,2}$ at the beginning of the turn	R_1	R_2	$Q_{1,2}$ at the end of the turn
3	1	1	3
3	2	1	4
3	1	2	2
3	2	2	3

The above are the same relative changes with the same probabilities as found for the case with $Q_{1,2} = 2$. It is reasonably easy to see that in general, when $Q_{1,2} = N \ \ (N \geq 2)$ at the start of a turn, the probabilities of other values at the end of a turn are always

$$P_{1,2}(N-1) = \frac{1}{4}$$

$$P_{1,2}(N) = \frac{1}{2}$$

$$P_{1,2}(N+1) = \frac{1}{4}$$

The case in the first turn where there is only one counter left at the end of the round has different outcomes in the next turn as player 2 will process one counter irrespective of the value of their roll, R_2.

$Q_{1,2}$ at the beginning of the turn	R_1	R_2	$Q_{1,2}$ at the end of the turn
1	1	1	1
1	2	1	2
1	1	2	1
1	2	2	2

So, in the case where $Q_{1,2} = 1$ at the start of a turn, the probabilities of other values at the end of a turn are

$$P_{1,2}(1) = \frac{1}{2}$$

$$P_{1,2}(2) = \frac{1}{2}$$

The values calculated are, in general, the probabilities that a value of $Q_{1,2} = N$ will change to a value of $Q_{1,2} = M$ in a turn. They are often called transition probabilities, and it is useful to arrange them as a matrix, T, where the transition probabilities are arranged as shown below.

$$\begin{array}{cc} & From\ Q_{1,2} = \\ & \begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \end{array} \\ To\ Q_{1,2} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Exercise 2: *Why must the columns of the transition matrix must sum to one.*

In addition to the above, a state vector after turn t, s_t is defined, where element i gives the probability $P_{1,2}(i)$ at each turn. Initially, $Q_{1,2} = 2$ with probability 1, i.e. $P_{1,2}(2) = 1$, while after one turn, the probabilities have been previously calculated to be $P_{1,2}(1) = \frac{1}{4}$, $P_{1,2}(2) = \frac{1}{2}$, $P_{1,2}(3) = \frac{1}{4}$. The state vectors for at turn t, s_t, are shown below.

$$s_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \qquad s_1 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \\ 0 \\ \vdots \end{bmatrix}$$

It is easy to show that the vector s_1 is given by the matrix-vector multiplication

$$s_1 = Ts_0$$

Interestingly, constructing the vector $s_2 = Ts_1$, gives the following

$$s_2 = \begin{bmatrix} \frac{1}{4} \\ \frac{7}{16} \\ \frac{1}{4} \\ \frac{1}{16} \\ 0 \\ \vdots \end{bmatrix}$$

Exercise 3: *Show that these are the correct probabilities of $Q_{1,2}$ having 1, 2, 3 or 4 counters after two turns.*

It can also be shown that

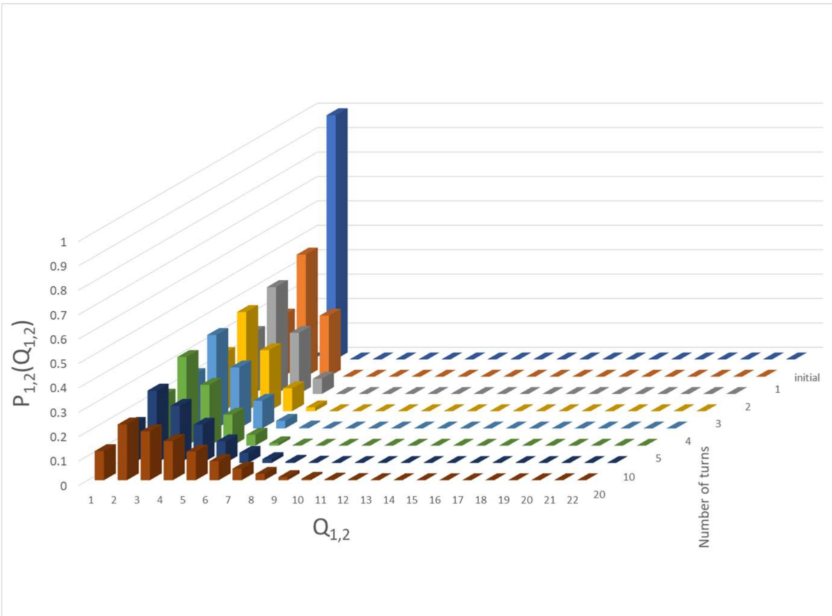
$$s_2 = Ts_1 = TTs_0 = T^2 s_0$$

And in general, after t turns

$$s_t = T^t s_0$$

Another general result is that after t turns, the last non-zero value in s_t is element $t + 2$. Therefore, the larger the number of turns, the larger the possible size of $Q_{1,2}$, i.e. the possible number of counters can grow without bound.

The evolution of the elements of the state vector with number of turns is shown below.



The approach used above can be used for the first two players in the actual game where die rolls can range from one to six. The number of possible roll combinations is now much larger: 6 x 6 rather than 2 x 2.

Exercise 4: *Show that when the initial value is $Q_{1,2} = 4$, the following probabilities are obtained for the value of $Q_{1,2}$ after a turn.*

$$P_{1,2}(1) = \frac{1}{12}$$

$$P_{1,2}(2) = \frac{1}{9}$$

$$P_{1,2}(3) = \frac{5}{36}$$

$$P_{1,2}(4) = \frac{1}{6}$$

$$P_{1,2}(5) = \frac{1}{6}$$

$$P_{1,2}(6) = \frac{1}{6}$$

$$P_{1,2}(7) = \frac{1}{12}$$

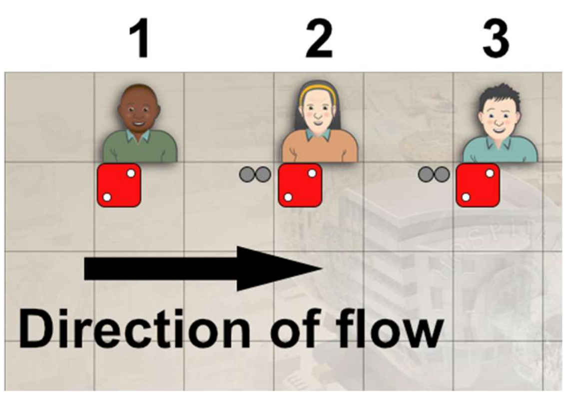
$$P_{1,2}(8) = \frac{1}{18}$$

$$P_{1,2}(9) = \frac{1}{36}$$

Repeating the above for all initial values of 1 to 6 allows the first six columns of the transition matrix to be constructed. The seventh column is the same as the sixth but with all elements one row lower, similar to the structure found in the simplified case analysed.

The first three players (simplified).

The next obvious step is to consider the first three players and the quantities $Q_{1,2}$ and $Q_{2,3}$, again using the simplified version where a die can roll a one or a two.



The pile $Q_{1,2}$ behaves as before, and it might be tempting to apply a similar approach to $Q_{2,3}$, but with a slight modification to account for the fact that while the change in $Q_{1,2}$ depends on the initial state of $Q_{1,2}$, and a random addition by player 1, the change in $Q_{2,3}$ depends not only on its initial state and a random addition by player 2, but also on the initial state of $Q_{1,2}$. If $Q_{1,2} \geq 2$, then the behaviour for $Q_{2,3}$ is as that for $Q_{1,2}$ and the transition matrix T can be used. However, if $Q_{1,2} = 1$, the transition matrix changes as if player 2 rolls a two then only one counter can be passed (a constraint that doesn't apply to player 1, who always brings in as many counters as the roll).

Exercise 5: *Show that in the case where $Q_{1,2} = 1$, the transition matrix, U, is given by*

$$\begin{array}{cc} & \textit{From } Q_{2,3} = \\ & \begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \end{array} \\ \textit{To } Q_{2,3} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} & \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

It is tempting to try to use the probability $P_{1,2}(1)$ to weight the use of the T matrix and the U matrix, so that the state vector for $Q_{2,3}$, after turn t, r_t is given by

$$r_t = P_{1,2}(1)Ur_{t-1} + \left(1 - P_{1,2}(1)\right)Tr_{t-1}$$

The logic being that the probability of $Q_{1,2}$ having one counter is $P_{1,2}(1)$, in which case the U transition matrix is used, while the probability of $Q_{1,2}$ having more than one counter is $\left(1 - P_{1,2}(1)\right)$, in which case the T transition matrix is used. However, beware – this does not work! To see why, consider the probabilities after one turn, then after two. As initially $Q_{1,2} = 2$, the T matrix is used in both cases and the state vectors are

$$s_1 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \\ 0 \\ \vdots \end{bmatrix}, \qquad r_1 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \\ 0 \\ \vdots \end{bmatrix}$$

These are the correct values. Applying $r_2 = P_{1,2}(1)Ur_1 + \left(1 - P_{1,2}(1)\right)Tr_1$ gives

$$r_2 = \begin{bmatrix} \frac{5}{16} \\ \frac{27}{64} \\ \frac{7}{32} \\ \frac{3}{64} \\ \vdots \end{bmatrix}$$

Which is wrong! The reason is that this method assumes that for every possible value of s_1 at the end of one turn, it is possible to use every possible value of r_1. To see why this is wrong consider all possible dice rolls and the values of $Q_{i,i+1}$ after the first round.

$Q_{1,2}$ at the beginning of the turn	$Q_{2,3}$ at the beginning of the turn	R_1	R_2	R_3	$Q_{1,2}$ at the end of the turn	$Q_{2,3}$ at the end of the turn
2	2	1	1	1	2	2
2	2	2	1	1	3	2
2	2	1	2	1	1	3
2	2	2	2	1	2	3
2	2	1	1	2	2	1
2	2	2	1	2	3	1
2	2	1	2	2	1	2
2	2	2	2	2	2	2

This gives the probabilities as shown in s_1 and r_1. Grouping outcomes another way

$Q_{2,3}$	Possible values of $Q_{1,2}$
1	2, 3
2	1, 2, 2, 3
3	1, 2

The above shows that you cannot have a state where $Q_{1,2} = 1$ *and* $Q_{2,3} = 1$, or a state where $Q_{1,2} = 3$ *and* $Q_{2,3} = 3$, as was naively assumed when using the expression

$$r_t = P_{1,2}(1)Ur_{t-1} + \left(1 - P_{1,2}(1)\right)Tr_{t-1}$$

When $Q_{2,3} = 1$ (which is ¼ of the outcomes of turn 1), $Q_{1,2}$ must be 2 or more, and the following probabilities after the next turn have been previously calculated for $P_{1,2}$ and are the same for $P_{2,3}$, i.e.

$$P_{2,3}(1) = \frac{1}{2}$$
$$P_{2,3}(2) = \frac{1}{2}$$

When $Q_{2,3} = 2$ (which is ½ of the outcomes of turn 1), $Q_{1,2}$ can be any value, but is twice as likely to be two than one or three. For the case where $Q_{1,2} = 1$ (which happens ¼ of the time when $Q_{2,3} = 2$) the probabilities have been calculated (when building the U matrix) as

$$P_{2,3}(1) = \frac{1}{2}$$
$$P_{2,3}(2) = \frac{1}{2}$$

When $Q_{2,3} = 2$ and $Q_{1,2} = 2$ or $Q_{1,2} = 3$ (which happens ¾ of the time when $Q_{2,3} = 2$) the probabilities have been previously calculated for $P_{1,2}$ and are the same for $P_{2,3}$, i.e.

$$P_{2,3}(1) = \frac{1}{4}$$

$$P_{2,3}(2) = \frac{1}{2}$$

$$P_{2,3}(3) = \frac{1}{4}$$

When $Q_{2,3} = 3$ (which is ¼ of the outcomes of turn 1), $Q_{1,2}$ must be 1 or 2 (½ the time each). When $Q_{1,2} = 1$, the probabilities calculated for the U matrix are used, i.e.

$$P_{2,3}(2) = \frac{1}{2}$$
$$P_{2,3}(3) = \frac{1}{2}$$

When $Q_{1,2} = 2$, the probabilities calculated for the T matrix are used, i.e.

$$P_{2,3}(2) = \frac{1}{4}$$
$$P_{2,3}(3) = \frac{1}{2}$$
$$P_{2,3}(4) = \frac{1}{4}$$

The total probabilities are calculated as a weighted sum of all components so that the following probability is found for the first element of state vector r_2.

$$P_{2,3}(1) = \underbrace{\frac{1}{4} \times \frac{1}{2}}_{Q_{2,3}=1} + \underbrace{\frac{1}{2} \times \frac{1}{4} \times \frac{1}{2}}_{\substack{Q_{2,3}=2 \\ Q_{1,2}=1}} + \underbrace{\frac{1}{2} \times \frac{3}{4} \times \frac{1}{4}}_{\substack{Q_{2,3}=2 \\ Q_{1,2}>1}} = \frac{9}{32}$$

Note, the above is slightly lower than the incorrect value of $\frac{5}{16}$ previously found. It is left as an exercise to calculate the value of $P_{2,3}(2)$, $P_{2,3}(3)$, $P_{2,3}(4)$ using the above to show that the correct state vector after two turns for $Q_{2,3}$ is

$$r_2 = \begin{bmatrix} \frac{9}{32} \\ \frac{15}{32} \\ \frac{7}{32} \\ \frac{1}{32} \\ \vdots \end{bmatrix}$$

At this point, it is becoming evident that the analysis is becoming quite complex, although it did give useful insight into how the size of $Q_{1,2}$ can grow as the number of turns assumed grows, and that in the long term, there is a non-zero probability that the queue will be, for the simplified case where the maximum roll, M, is 2 and $Q_0 = 2$, two more than the number of turns taken. For the more general case where M and Q_0 can take arbitrary values, it can be shown that the maximum theoretical possible number of counters in $Q_{1,2}$ after t turns, $Z(t, Q_0)$, and assuming the minimum roll value is 1, is given by

$$Z(t, Q_0) = (M - 1)t + Q_0$$

Note, the above expression is useful when constructing a computer simulation of the system.

Exercise 6: *Derive the above expression for Z.*

As a last point, the analysis only considers the number of counters between players and not the throughput. The actual throughput at the end of the game can be obtained by adding an extra dummy player that always rolls a zero, i.e. the number of counters between the last actual player and the dummy player can only ever increase. Its value at the end of the game gives the throughput of the system. It is left as an exercise to incorporate this into the analysis.

Further analytic development is left as a research exercise.

Simulation

Another possible approach is computer simulation. For small problems the simulation can consider every possible outcome explicitly, while for larger problems we must be content with a random sample of games, which should give results that tend towards the actual values as the sample size increases.

A definition of "small problem" by enumerating the possible number of games playable. This depends on the number of players, p, the number of roll outcomes on a die, V, and the number of turns played, t, and is given by

$$\Omega = V^{tp}$$

For 3 players, using die that can roll 1 or 2 and 5 turns, there are

$$\Omega = V^{tp} = 2^{5\times 3} = 2^{15} = 32{,}768$$

possible games. This is easily covered in its entirety with a computer simulation. For 10 players, using die that can roll 1 to 6, and 20 turns, there are

$$\Omega = V^{tp} = 6^{20\times 10} = 6^{200} \sim 4 \times 10^{155}$$

possible games. This is an unimaginably large number! (A lot of the games will be degenerate in their outcome, so that the total number actually required is fewer than this, but significant effort is required to identify the degeneracies!)

An example code written in Python 3 that uses sampling is appended to this document. It can be set up to run for a number of different problems by changing the values of the game configuration variables:

```
players = 3
rounds = 2
startQ = 2
dieLow = 1
dieHigh = 2

sampleSize = 1000000
```

The first two are evident (you must use at least two players!). The value of startQ defines the initial number of counters between players, while dieLow and dieHigh give the minimum and maximum dice roll values. The value of sampleSize determines how many games are played to generate statistics.

The above configuration determines the state vectors for $Q_{1,2}$ and $Q_{2,3}$ after two rounds in the simplified game with three players. It produces output similar to

```
  0% complete
 10% complete
 20% complete
 30% complete
 40% complete
 50% complete
 60% complete
 70% complete
 80% complete
 90% complete
 Finished

Number of players: 3
Number of rounds:  2
Sample size:       1,000,000

        Q 1 ,2       Q 2 ,3
P( 1) 2.497590E-01 2.806890E-01
P( 2) 4.379210E-01 4.689550E-01
P( 3) 2.498410E-01 2.191390E-01
P( 4) 6.247900E-02 3.121700E-02
```

These may be compared with the analytic values

$$s_1 = \begin{bmatrix} \frac{1}{4} \\ \frac{7}{16} \\ \frac{1}{4} \\ \frac{1}{16} \\ \vdots \end{bmatrix} = \begin{bmatrix} 2.500000E-01 \\ 4.375000E-01 \\ 2.500000E-01 \\ 6.250000E-02 \\ \vdots \end{bmatrix} \qquad r_2 = \begin{bmatrix} \frac{9}{32} \\ \frac{15}{32} \\ \frac{7}{32} \\ \frac{1}{32} \\ \vdots \end{bmatrix} = \begin{bmatrix} 2.812500E-01 \\ 4.687500E-01 \\ 2.187500E-01 \\ 3.125000E-02 \\ \vdots \end{bmatrix}$$

The values are reasonably close, but a larger sample is required to improve the agreement. Change the sample size and observe how the calculated values tend towards the actual values. Change the other game configuration variables to simulate other game set-ups. The full dice game is simulated with

```
players = 10
rounds = 20
startQ = 4
dieLow = 1
dieHigh = 6
```

As with the analysis, the simulation only considers the number of counters between players and not the throughput. The actual throughput at the end of the game can be obtained by adding an extra dummy player that always rolls a zero, i.e. the number of counters between the last actual player and the dummy player can only ever increase. Its value at the end of the game gives the throughput of the system. It is left as an exercise to incorporate this into the simulation.

The following Python code simulates the game. The text is in a box to show where the left tab positions are as this is critical to Python. Note, Python is an interpreted language and long calculation times can be encountered for large sample sizes. It is reasonably easy to convert the code to a compiled language such as C++, Java, FORTRAN to produce results in a shorter time.

```python
# -*- coding: utf-8 -*-
"""
Dice Game Simulation Base Code

@author: Chris Robbins, Grallator

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subject to this whole attribution section being included.

The code is supplied as-is with no warranty or guarantees.

"""

import random


outcome = []
players = 3
rounds = 2
sampleSize = 1000000

startQ = 2

# note, python arrays are indexed from 0 so there will be a lot of A[x+1] and A[x-1] to
# keep the index in line with the player and queue numbers

# Q[0] is between players 1 and 2, etc
Q=[]

# R[0] is the die roll of player 1, etc
R=[]

dieLow = 1
dieHigh = 2

#
# stateVector[0] is the probability of one counter etc for each Q
# these are stacked inside outcome so that, for example:
# outcome[0][0] is the probability of one counter at Q[0], i.e. between player 1 and 2
# outcome[0][1] is the probability of two counters at Q[0], i.e. between player 1 and 2
# outcome[1][0] is the probability of one counters at Q[1], i.e. between player 2 and 3
# etc
for i in range(players):
    outcome.append([])
    for j in range((dieHigh-1)*rounds+startQ):
        outcome[i].append(0)

    if i<players-1:
        Q.append(startQ)

    R.append(0)
```

```
# start working with a random sample
for sampleItem in range(sampleSize):
    # give some indication that the calculation is progressing
    if (sampleItem%(sampleSize/10)==0):
        print("{:3.0f}% complete".format(100*sampleItem/sampleSize))

    # new game - initialise the values in Q
    #
    for q in range(players-1):
        Q[q]=startQ

    # play the game for the given number of rounds
    for r in range(rounds):
        # the roll round
        for p in range(players):
            # roll the die for player p+1
            # the Python random number generator is based on the he Mersenne Twister
            # and has a period of 2**19937-1 - so is a good choice!
            R[p]=random.randint(dieLow, dieHigh)

            # the first player always moves a number of counters equal to the roll
            # into the system. However, other players can only move the number of
            # counters they have. So first check if this is not the first player
            if p>0:
                # Q[p-1] is the number of counters the player can process. If this
                # is less than the actual roll, reduce the roll value to match so
                # that the value of Q[] does not go negative
                if R[p]>Q[p-1]:
                    R[p]=Q[p-1]

        # the move round
        for q in range(players-1):
            Q[q]=Q[q]-R[q+1]+R[q]

    # end of game - process the stats
    for q in range(players-1):
        # add the Q size for each Q to the appropriate element in outcome
        outcome[q][Q[q]-1]=outcome[q][Q[q]-1]+1


# finished taking sample - scale the results and print them

print(" Finished")
print()
print("Number of players:", players)
print("Number of rounds: ", rounds)
print("Sample size:       {:,.0f}".format(sampleSize))
print()
outLine="     "
for q in range(players-1):
    outLine=outLine+"   Q {:<2},{:<2}    ".format(q+1,q+2)
print(outLine)

for n in range(rounds+2):
    outLine="P({:2.0f})".format(n+1)
    for q in range(players-1):
        outcome[q][n]=outcome[q][n]/sampleSize
        outLine=outLine+"{:13.6E}".format(outcome[q][n])
    print(outLine)
```

Answers to exercises

Exercise 1 The initial condition is that $Q_{i,i+1} > 0 \ \ \forall\, Q_{i,i+1}$ and the die roll has a range $1 \leq R_i \leq M$, for a maximum roll value M. Player 1 always adds $R_1 \geq 1$ counters to the system. The value of R_2 may be such that player 2 can process all the counters in the pile $Q_{1,2}$, however $R_1 \geq 1$ counters will be added by player 1, which cannot be removed by player 2 *in the same round*. Therefore $Q_{1,2}$ always has at least one counter at the end of a round. As $Q_{1,2}$ always has at least one counter, even in the first round as $Q_{1,2}$ is initialised with $Q_{1,2} > 1$, there will always be at least one counter available to be passed by player 2 to $Q_{2,3}$ in any round. By induction there will always be at least one counter to pass to any $Q_{i,i+1}$ in any turn, and therefore the minimum value of $Q_{i,i+1}$ at any time is 1.

Exercise 2 The columns in the transition matrix give the probability of the value of $Q_{i,i+1}$ equal to the column index at the start of the turn becoming the value of the row index at the end of the turn. If the matrix has been constructed correctly, then all possibilities have been accounted for and the sum of all the probabilities must sum to one. Therefore, the sum of entries in a column must sum to one.

Exercise 3 After the first round the following probabilities for the value of $Q_{1,2}$ are found:

$$P_{1,2}(1) = \tfrac{1}{4}$$

$$P_{1,2}(2) = \tfrac{1}{2}$$

$$P_{1,2}(3) = \tfrac{1}{4}$$

For the case where $Q_{1,2} = 1$, which occurs with probability ¼, the following outcomes probabilities have been calculated:

$$P_{1,2}(1) = \tfrac{1}{2}$$

$$P_{1,2}(2) = \tfrac{1}{2}$$

For the case where $Q_{1,2} = 2$, which occurs with probability ½, the following outcomes probabilities have been calculated:

$$P_{1,2}(1) = \tfrac{1}{4}$$

$$P_{1,2}(2) = \tfrac{1}{2}$$

$$P_{1,2}(3) = \tfrac{1}{4}$$

For the case where $Q_{1,2} = 3$, which occurs with probability ¼, the following outcomes probabilities have been calculated:

$$P_{1,2}(2) = \frac{1}{4}$$

$$P_{1,2}(3) = \frac{1}{2}$$

$$P_{1,2}(4) = \frac{1}{4}$$

The total probability is found by a weighted sum of all the individual probabilities (sum of probability starting Q is given value multiplied by probability the starting Q gives the required Q value at the end of the turn):

$$P_{1,2}(1) = \underbrace{\frac{1}{4} \times \frac{1}{2}}_{\substack{Q_{1,2}=1 \\ after \\ first \\ round}} + \underbrace{\frac{1}{2} \times \frac{1}{4}}_{\substack{Q_{1,2}=2 \\ after \\ first \\ round}} = \frac{1}{4}$$

$$P_{1,2}(2) = \underbrace{\frac{1}{4} \times \frac{1}{2}}_{\substack{Q_{1,2}=1 \\ after \\ first \\ round}} + \underbrace{\frac{1}{2} \times \frac{1}{2}}_{\substack{Q_{1,2}=2 \\ after \\ first \\ round}} + \underbrace{\frac{1}{4} \times \frac{1}{4}}_{\substack{Q_{1,2}=3 \\ after \\ first \\ round}} = \frac{7}{16}$$

$$P_{1,2}(3) = \underbrace{\frac{1}{2} \times \frac{1}{4}}_{\substack{Q_{1,2}=2 \\ after \\ first \\ round}} + \underbrace{\frac{1}{4} \times \frac{1}{2}}_{\substack{Q_{1,2}=3 \\ after \\ first \\ round}} = \frac{1}{4}$$

$$P_{1,2}(4) = \underbrace{\frac{1}{4} \times \frac{1}{4}}_{\substack{Q_{1,2}=3 \\ after \\ first \\ round}} = \frac{1}{16}$$

Exercise 4 For this case it is useful to construct an outcome table that shows the outcome value of $Q_{1,2}$ for the case where $Q_0 = 4$, and for every possible combination of rolls for player 2, removing up to but not more than four counters, and player 1, subsequently adding that number of counters, as shown below.

$Q_0 = 4$		Roll R_2					
		1	**2**	**3**	**4**	**5**	**6**
Roll R_1	**1**	4	3	2	1	1	1
	2	5	4	3	2	2	2
	3	6	5	4	3	3	3
	4	7	6	5	4	4	4
	5	8	7	6	5	5	5
	6	9	8	7	6	6	6

The probabilities $P_{1,2}(N)$ are simply given by counting up the fractions for each value of N from 1 to 9. This gives, for example

$$P_{1,2}(1) = \frac{3}{36} = \frac{1}{12}$$

$$P_{1,2}(2) = \frac{4}{36} = \frac{1}{9}$$

etc., as required.

Exercise 5

In this problem $Q_{1,2} = 1$, so only one counter can ever be transferred to $Q_{2,3}$, regardless of the value of player 2's roll, R_2. For the case where $Q_{2,3} = 1$ at the start of the turn, player 3 will always roll a value $R_3 \geq 1$ and so will always process all the counters. The outcome is that player 3 always empties $Q_{2,3}$ while player 2 always adds one counter to it, so that $Q_{2,3} = 1$ always, i.e. with probability 1. This means the transition probability of going from $Q_{2,3} = 1$ to $Q_{2,3} = 1$ is 1

When $Q_{2,3} = N \geq 2$, player 3 can remove one or two counters, each with probability ½. When Player 2 adds the counter they have, the net result is that ½ the time $Q_{2,3} = N$ after the turn (one out, one in), and ½ the time $Q_{2,3} = N - 1$ after the turn (two out, one in). So the transition probabilities are

Probability of going from $Q_{2,3} = N$ to $Q_{2,3} = N - 1$ is ½.
Probability of going from $Q_{2,3} = N$ to $Q_{2,3} = N$ is ½.

Using these results gives the transition matrix U as required.

Exercise 6

Initially $Q_{1,2} = Q_0$. The maximum value after the first round is given when $R_2 = 1$, i.e. the minimum number of counters possible is processed and removed, and $R_1 = M$, i.e. the maximum number of counters possible is added. In this case, after the turn

$$Q_{1,2} = Q_0 - 1 + M$$

Which reads: start with Q_0, process 1, add in M. I'll take the liberty of rearranging the above to the following

$$Q_{1,2} = M - 1 + Q_0$$

On the second turn, the maximum possible number of counters is given when the same roll outcome is encountered (i.e. one is processed and M are added) so that the maximum value after two turns is

$$Q_{1,2} = M - 1 + Q_0 + M - 1 = 2(M - 1) + Q_0$$

After each turn a further $(M - 1)$ can be added so that after t turns, the maximum possible size of $Q_{1,2}$, denoted Z, is

$$Z = (M - 1)t + Q_0$$